Comment on "Heat fluctuations in Brownian transducers"

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The results presented in [A. Gomez-Marin and J.M. Sancho, Phys. Rev. E **73**, 045101(R) (2006)] are mathematically incomplete and physically faulty. Contrary to their claim, there exists a fluctuation theorem for the heat probability distribution function in Brownian transducers operating between two heat reservoirs. The corresponding large deviation function is determined exactly.

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The system considered in Ref. [1] can be modeled by a set of coupled Langevin equations for two degrees of freedom x and y

$$\frac{dx}{dt} = -k(x-y) + \xi_1, \quad \frac{dy}{dt} = -k(y-x) + \xi_2, \tag{1}$$

where ξ_i is a Gaussian white noise with variance $2T_i$ and $k>0$ is a coupling constant. The heat $Q(t)$ outcoming from the reservoir at T_2 over a time duration t that is the subject of Ref. [1], can be written, using the Stratonovitch discretization prescription, in the form

$$Q(t) = -k\int_0^t dt r(t)\frac{dy}{dt} = \int_0^t dt(k^2r^2 - kr\xi_2), \tag{2}$$

where $r(t)=y(t)-x(t)$ can be found by solving Eq. (1) and has the expression $r(t)=\int^t d\tau e^{-2k(t-\tau)}[\xi_2(\tau)-\xi_1(\tau)]$. Therefore Q as defined in Eq. (2) is a quadratic form in the noises ξ_1 and ξ_2, and evaluating the average $\langle e^{-\lambda Q}\rangle$ with respect to the Gaussian noises amounts to performing a simple Gaussian integral (see Refs. [2–4] for similar calculations). In the Fourier representation, the matrix of the quadratic form in ξ_1,ξ_2 reads

$$\Gamma(\omega,\lambda) = \begin{pmatrix} \frac{\beta_1}{2} + \frac{2\lambda k^2}{\omega^2+4k^2} & -\frac{2\lambda k^2}{\omega^2+4k^2} + \frac{\lambda k}{i\omega+2k} \\ -\frac{2\lambda k^2}{\omega^2+4k^2} + \frac{\lambda k}{-i\omega+2k} & \frac{\beta_2}{2} - \frac{2\lambda k^2}{\omega^2+4k^2} \end{pmatrix}. \tag{3}$$

The generating function of the cumulants of $Q(t)$, denoted by $\mu(\lambda)$, and defined by $\mu(\lambda)=\lim_{t\to\infty}\frac{\ln\langle e^{-\lambda Q}\rangle}{t}$ is obtained from the determinant of $\Gamma(\omega,\lambda)$

$$\mu(\lambda) = \frac{1}{2}\int\frac{d\omega}{2\pi}\ln\frac{\det\Gamma(\omega,0)}{\det\Gamma(\omega,\lambda)} = \frac{1}{2}\int\frac{d\omega}{2\pi}\ln\frac{\omega^2+4k^2}{\omega^2+4k^2(1-\lambda T_1)(1+\lambda T_2)} \tag{4}$$

and is explicitly given by

$$\mu(\lambda) = k[1-\sqrt{(1-\lambda T_1)(1+\lambda T_2)}]. \tag{5}$$

Of course, the first and second cumulants of Q given, respectively, by (minus) the first and the second derivatives of $\mu(\lambda)$ are identical to the results presented in Ref. [1]. It is straightforward to see that μ possesses an important symmetry property, namely,

$$\mu(\lambda) = \mu(T_1^{-1} - T_2^{-1} - \lambda). \tag{6}$$

In order to return to the quantity of interest $Q(t)$ and to its large deviation function $\pi(q)$ defined in terms of the fluctuating time-averaged heat $q(t)=\frac{Q(t)}{t}$ by

$$\pi(q) = \lim_{t\to\infty}\frac{1}{t}\ln P(Q=qt,t) \tag{7}$$

we use the fact that $\pi(q)$ is simply the Legendre transform of $\mu(\lambda)$, $\pi(q)=\max_\lambda\{\lambda q+\pi(q)\}$. Given that $\mu(\lambda)$ is not quadratic in λ, $\pi(q)$ is not quadratic either and, in fact, $P(Q,t)\sim e^{t\pi(Q/t)}$ is far from a Gaussian. The symmetry property (6) therefore leads to

$$\pi(q) - \pi(-q) = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)q. \tag{8}$$

This is the celebrated fluctuation theorem [5,6]. A few words of caution must be added. It is by now well documented [3,4,7] that the fluctuation theorem (8) can be spoiled for $|q|$ beyond a certain model-dependent—but finite—value q^*, somewhat restricting the theorem to a finite range of q. The precise value of q^* depends on how the system is prepared initially (see Ref. [4] for a recent discussion).

In conclusion, the approximations carried out in Ref. [1] are unphysical. They violate the fluctuation theorem that, as we have shown, holds for the model studied by these authors.

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